New Numerical Algorithms for Unconstrained Optimization Problems

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ABSTRACT

Two new algorithms namely, Circle approach algorithm and Circle-tangent approach algorithm are proposed for unconstrained optimization problems. Then, comparative study among the new algorithms and Newton’s algorithm are established by means of examples.

1. Introduction

Optimization problems with or without constraints arise in various fields such as science, engineering, economics, management sciences, etc., where numerical information is processed. In recent times, many problems in business situations and engineering designs have been modeled as an optimization problem for taking optimal decisions. In fact, numerical optimization techniques have made deep in to almost all branches of engineering and mathematics.

An unconstrained minimization problem is the one where a value of the vector $x$ is sought that minimizes the objective function $f(x)$. This problem can be considered as particular case of the general constrained non-linear programming problem. The study of unconstrained minimization techniques provide the basic understanding necessary for the study of constrained minimization methods and this method can be used to solve certain complex engineering analysis problem. For example, the displacement response (linear or non-linear) of any structure under any specified load condition can be found by minimizing its potential energy.

Several methods [5,7,8,10,11,15,16] are available for solving unconstrained minimization problems. These methods can be classified in to two categories as non gradient and gradient methods. The non gradient methods require only the objective function values but not the derivatives of the function in finding minimum. The gradient methods require, in addition to the function values, the first and in some cases the second derivatives of the objective function. Since more information about the function being minimized is used through the use of derivatives, gradient methods are generally more efficient than non gradient methods. All the
unconstrained minimization methods are iterative in nature and hence they start from an initial trial solution and proceed towards the minimum point in a sequential manner.


In this paper, we propose two new algorithms namely, Circle approach algorithm and Circle-tangent approach algorithm for minimizing nonlinear real valued and twice differentiable real functions. Then, we present the comparative study among the new algorithms and Newton’s algorithm by means of examples.

2. New algorithms

In this section, we introduce two new numerical algorithms namely, circle approach algorithm and circle-tangent approach algorithm for minimizing nonlinear real valued and twice differentiable real functions using the concept of external touch numerical algorithm [18].

Consider the nonlinear optimization problem:
Minimize \( \{ f(x), \ x \in \mathbb{R}, \ f : \mathbb{R} \to \mathbb{R} \} \)
where \( f \) is a non-linear twice differentiable function.

2.1 Circle approach algorithm

Consider the function \( G(x) = x - (g(x)/g'(x)) \) where \( g(x) = f'(x) \).
Here \( f(x) \) is the function to be minimized. \( G'(x) \) is defined around the critical point \( x^* \) of \( f(x) \) if \( g'(x^*) = f''(x^*) \neq 0 \) and is given by
\[
G'(x) = g(x)g''(x)/g'(x).
\]
If we assume that \( g'(x^*) \neq 0 \), we have \( G'(x^*) = 0 \) iff \( g(x^*) = 0 \).

Consider the equation
\[
g(x) = 0
\]
whose one or more roots are to be found. \( y = g(x) \) represents the graph of the function \( g(x) \) and assume that an initial estimate \( x_0 \) is known for the desired root of the equation \( g(x) = 0 \).

A circle \( C_1 \) of radius \( g(x_0 + h) \) is drawn with centre at any point \( (x_0 + h, g(x_0 + h)) \) on the curve of the function \( y = g(x) \) where \( h \) is small positive or negative quantity. Another circle \( C_2 \) with radius \( g(x_0 - h) \) and centre at \( (x_0 - h, g(x_0 - h)) \) is drawn on the curve of the function \( g(x) \) such that it touches the circle \( C_1 \) externally.

Let \( x_1 = x_0 + h \), \( |h| < 1 \).

Since the circles \( C_2 \) and \( C_1 \) touches externally, we have
\[
h^2 = g(x_0 + h)g(x_0 - h).
\]
Expanding \( g(x_0 + h) \) and \( g(x_0 - h) \) by Taylor’s series (omitting fourth and higher powers of \( h \)) and simplifying, we can conclude that
\[
h = \pm \frac{g(x_0)}{\sqrt{1 + g'(x_0)^2 - g(x_0)g''(x_0)}}
\]
where \( h \) can be taken positive or negative according as \( x_0 \) lies in the left or right of true root or slope of the curve at \( (x_0, g(x_0)) \) is positive or negative. If \( x_0 \) lies in the left of true root, then \( h \) is taken as positive otherwise negative. Therefore, we get the first approximation to the root as \( x_1 = x_0 \pm h \).

That is, \( x_1 = x_0 \pm \frac{g(x_0)}{\sqrt{1 + g'(x_0)^2 - g(x_0)g''(x_0)}} \).

Since \( g(x) = f'(x) \), it follows that \( x_1 = x_0 \pm \frac{f''(x_0)}{\sqrt{1 + f'(x_0)^2 - f'(x_0)f'''(x_0)}} \).

The general iteration formula for successive approximate minimizing point of the non-linear function \( f' \) is
\[
x_{n+1} = x_n \pm \frac{f''(x_n)}{\sqrt{1 + f''(x_n)^2 - f''(x_n)f'''(x_n)}}
\]

----(3)
Note: The sufficient condition for convergence in the interval containing the root is given \( f''(x_n) f'''(x_n) < 1 + f''(x_n) \)

Now, we give the Circle approach algorithm for finding the minimizing point of the non-linear function of the real valued real functions

**Algorithm:**

1. Given a non-linear function \( f(x) \)
2. Find \( a \) and \( b \) such that \( f'(a) \) and \( f'(b) \) are of opposite signs such that \( a < b \).
3. Input \( x_0, \varepsilon, f'(x), f''(x), f'''(x) \)
4. \( n = 0 \)
5. If we choose \( x_0 = a \), then go to step 6
   otherwise if we choose \( x_0 = b \), then go to step 12
6. Repeat
7. \( x_{n+1} = x_n + \frac{f'(x_n)}{\sqrt{1 + f''(x_n) - f'(x_n) f'''(x_n)}} \)
8. \( n \leftarrow n + 1 \)
9. Until \( |x_n - x_{n-1}| < \varepsilon \)
10. Optimal solution \( x^* \leftarrow x_n \)
11. End
12. Repeat
13. \( x_{n+1} = x_n - \frac{f'(x_n)}{\sqrt{1 + f''(x_n) - f'(x_n) f'''(x_n)}} \)
14. \( n \leftarrow n + 1 \)
15. Until \( |x_n - x_{n-1}| < \varepsilon \)
16. Optimal solution \( x^* \leftarrow x_n \)
17. End

**Convergence analysis for Circle approach algorithm**

This algorithm is similar to the external touch technique of circles [18]. Since the order of convergence of the external touch technique of circles is 2, we have the order of convergence this method is 2.

By Newton’s algorithm and Circle approach algorithm, a minimizing point for the given function are obtained and then, the comparative study of the above said algorithms have been established by means of examples.
Example 1: Consider the function $f(x) = x^4 - x - 10 , x \in R$. Then, minimizing point of the function is equal to 0.62996 which is obtained in 6 iterations by Circle approach algorithm and in 8 iterations by Newton’s algorithm (refer Table -1 and Figure-1).

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Newton’s algorithm</th>
<th>Circle approach algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
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<tr>
<td>2</td>
<td>2.00926</td>
<td>1.299873</td>
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<td>3</td>
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<td>0.62999</td>
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<tr>
<td>6</td>
<td>0.64223</td>
<td>0.62996</td>
</tr>
<tr>
<td>7</td>
<td>0.63019</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.62996</td>
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</tbody>
</table>

Figure 1. Series B depicts Newton’s algorithm and series C depicts Circle approach algorithm.

Example 2: Consider the function $f(x) = xe^x - 1 , x \in R$. Then, minimizing point of the function is equal to -1 which is obtained in 10 iterations by Newton’s algorithm and in 18 iterations by Circle approach algorithm (refer Table -2 and Figure-2).

Remark: From the above examples, we observe that the minimizing sequential points of the given function obtained by Circle approach algorithm converges faster than Newton’s algorithm to a very near point of the minimizing point initially.
and from that point to the actual minimizing point of the function it converges slowly or equally as Newton’s algorithm.

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<tr>
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<td>1</td>
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<tr>
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</tr>
<tr>
<td>4</td>
<td>0.728954</td>
<td>-0.99789</td>
</tr>
<tr>
<td>5</td>
<td>0.095395</td>
<td>-0.99862</td>
</tr>
<tr>
<td>6</td>
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<td>-0.9991</td>
</tr>
<tr>
<td>7</td>
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<tr>
<td>9</td>
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<td>-0.99975</td>
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<tr>
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<tr>
<td>11</td>
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<td>-0.99989</td>
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<td>12</td>
<td>-1</td>
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<td>13</td>
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<td>-0.99999</td>
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<tr>
<td>18</td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

**Figure 2.** Series B depicts Newton’s algorithm and series C depicts Circle approach algorithm

2.2 Circle-tangent approach algorithm
Now, we propose another new algorithm namely, Circle-tangent approach algorithm which is the combination of Circle approach algorithm and Newton’s algorithm such that Circle approach algorithm is used initially for one or two iterations and then, Newton’s algorithm is used for obtaining the minimizing point of the function using the approximate minimizing point by the Circle approach algorithm. The algorithm of Circle-tangent approach is given below.

**Algorithm:**

1. Given a non-linear function \( f(x) \)
2. Find \( a \) and \( b \) such that \( f'(a) \) and \( f'(b) \) are of opposite signs such that \( a < b \).
3. Input \( x_0, \epsilon, f'(x), f''(x), f'''(x) \)
4. \( n = 0 \)
5. If we choose \( x_0 = a \), then go to step 6
   otherwise if we choose \( x_0 = b \) then go to step 12.
6. Repeat
7. For \( n = 1 \), compute
   \[
   x_{n+1} = x_n + \frac{f'(x_n)}{\sqrt{1 + f''(x_n) - f'(x_n)f'''(x_n)}}
   \]
8. For \( n = 2, 3, 4, \ldots \), compute \( x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} \)
9. Until \( |x_n - x_{n-1}| < \epsilon \)
10. Optimal solution \( x^* \leftarrow x_n \)
11. End
12. Repeat
13. For \( n = 1 \), compute
   \[
   x_{n+1} = x_n - \frac{f'(x_n)}{\sqrt{1 + f''(x_n) - f'(x_n)f'''(x_n)}}
   \]
14. For \( n = 2, 3, 4, \ldots \), compute \( x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} \)
15. Until \( |x_n - x_{n-1}| < \epsilon \)
16. Optimal solution \( x^* \leftarrow x_n \)
17. End

**Convergence analysis for Circle tangent algorithm**

This algorithm is a combination of the two algorithms namely, Circle approach algorithm and Newton’s algorithm. For \( n = 1 \), we use Circle approach algorithm and for \( n = 2, 3, 4, \ldots \), we use Newton’s algorithm. Since the order of
A minimizing point for a given function by Newton’s algorithm, Circle approach algorithm and Circle-tangent approach algorithm are obtained and the comparative study of the Newton’s algorithm, Circle approach algorithm and Circle-tangent approach algorithm have been established by means of examples.

Example 3: Consider the function \( f(x) = xe^{x} - 1 \), \( x \in \mathbb{R} \). Then, minimizing point of the function is equal to \(-1\) which is obtained by Newton’s algorithm in 10 iterations, Circle approach algorithm in 18 iterations and Circle-tangent approach algorithm in 4 iterations (refer Table-3 and Figure-3).

### Table -3

<table>
<thead>
<tr>
<th>Iterations</th>
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<tbody>
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3. Conclusion

In this paper, we have introduced new numerical algorithms namely, Circle approach algorithm and Circle-tangent approach algorithm for minimizing nonlinear unconstrained optimization problems. In real life problems, the variables can not be chosen arbitrarily rather they have to satisfy certain specified conditions called constraints. Such problems are known as constrained optimization problems. In near future, we have a plan to extend the proposed new algorithms namely, Circle
approach algorithm and Circle-tangent approach algorithm to constrained optimization problems.

**Figure 3.** Series B depicts Newton’s algorithm, series C depicts Circle approach algorithm and series D depicts Circle-tangent algorithm

**REFERENCES**